Hilbert schemes of a surface and Euler characteristics

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Abstract

We use basic algebraic topology and Ellingsrud-Stromme results on the Betti numbers of punctual Hilbert schemes of surfaces to compute a generating function for the Euler characteristic numbers of the Douady spaces of "n-points" associated with a complex surface. The projective case was first proved by L. Göttsche.

0   Introduction

This note is dedicated to the determination of the Euler characteristic numbers $e(X^{[n]})$ of the Douady spaces of "n-points" $X^{[n]}$ associated with a complex surface X; see Theorem 2.1.

For X algebraic (the Douady space is replaced by the Hilbert scheme) see [9], [10] and [4]. In the projective case L. Göttsche [9] used P. Deligne's solution to the Weil Conjectures and Ellingsrud-Stromme's determination of the Betti numbers of punctual Hilbert schemes [7]. Later, Göttsche-Soergel [10] proved the quasi-projective case using the methods of perverse sheaves. J. Cheah [4] obtained similar results using the theory of Mixed Hodge structures and [7].

The proof presented here, valid without the algebraicity restriction on X, uses [7] and elementary algebraic topology.

1   Notation and preliminaries

The topological Euler characteristic of a topological space Y, with respect to singular cohomology with compact supports, is denoted by $e(Y)$. If a statement involves this quantity $e(Y)$, it is understood that part of the statement consists of the assertion that $e(Y)$ is defined.

The term *complex surface* denotes, here, a smooth, connected and complex analytic surface with separable topology.

Let X be a complex surface and $n \in \mathbb{N}^*$. Denote by $X^{[n]}$ the Douady Space [6] of zero-dimensional analytic subspaces of X of length n; in the algebraic case this notion is replaced by the one of the Hilbert scheme; note that the "local differential geometries" of the Hilbert scheme and of the Douady spaces run parallel so that one can translate most of the proofs in [8] in the analytic context. The spaces $X^{[n]}$ are connected, complex manifolds of dimension $2n$. The corresponding Barlet space (cf. [1]) of zero-dimensional cycles in X of total multiplicity n can be identified with the symmetric product $X^{(n)} = X^n/\Sigma_n$, where Σ_n denotes the symmetric group over n elements and it acts by permuting the

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factors of X^n. There is a natural proper holomorphic map $\pi : X^{[n]} \to X^{(n)}$ which is a resolution of the singularities of $X^{(n)}$ (cf. [1]); we call this morphism the Douady-Barlet morphism.

There is a natural stratification for this morphism given by the partitions of n. Let $P(n)$ denote the set of partitions of the natural number n:

$$P(n) := \left\{ \nu = (\nu_1, \ldots, \nu_k) \mid k \in \mathbb{N}^*, \ \nu_j \in \mathbb{N}^*, \ \nu_1 \geq \nu_2 \geq \ldots \geq \nu_k > 0, \sum_{j=1}^{k} \nu_j = n \right\}.$$

The cardinality of $P(n)$ is denoted by $p(n)$. Every partition $\nu = (\nu_1, \ldots, \nu_k)$ can also be represented by a sequence of n nonnegative integers:

$$\nu = (\nu_1, \ldots, \nu_k) \Longleftrightarrow \{\alpha_1, \ldots, \alpha_n\},$$

where α_i is the number of times that the integer i appears in the sequence $(\nu_1, \ldots, \nu_k)$. Clearly $n = \sum_{i=1}^{n} i\,\alpha_i$. The number k associated with ν is called *the length of ν* and it is usually denoted by $\lambda(\nu)$. The set $P(n)$ is in natural one to one correspondence with a set of mutually disjoint subsets of $X^{(n)}$ which are locally closed for the Zariski topology on analytic spaces: associate with every $\nu \in P(n)$ the space

$$S_\nu^n X = S_{(\nu_1, \ldots, \nu_k)}^n X := \left\{ \sum_{j=1}^{k} \nu_j x_j \in X^{(n)} \mid x_{j_1} \neq x_{j_2} \ if \ j_1 \neq j_2 \right\}.$$

The space $X^{(n)}$ is the disjoint union of these locally closed subsets:

$$X^{(n)} = \coprod\nolimits_{\nu \in P(n)} S_\nu^n X. \tag{1}$$

Define $\widetilde{S_\nu^n X} := \pi^{-1}(S_\nu^n X)$. The space $X^{[n]}$ is the disjoint union of these locally closed subsets:

$$X^{[n]} = \coprod\nolimits_{\nu \in P(n)} \widetilde{S_\nu^n X}. \tag{2}$$

Given $m \in \mathbb{N}^*$, the analytic isomorphism class of the fiber over a point $x \in X = S_{(m)}^m X$ of the Douady-Barlet morphism $\pi : X^{[m]} \to X^{(m)}$ are independent of the surface X and of the point $x \in X$; since they are isomorphic to the fiber in question when X is the affine plane $\mathbb{A}^2 = \mathbb{C}^2$ and the point x is the origin, we denote them by $\mathbb{A}_{[0]}^{2^{[m]}}$.

Theorem 1.1 *(Cf. [7]) $e\left(\mathbb{A}_{[0]}^{2^{[m]}} \right) = p(m)$.*

Results of Fogarty's [8] (see also [9]), adapted in this analytic context, ensure that the restriction of π, $\pi : \widetilde{S_\nu^n X} \to S_\nu^n X$, is a locally trivial fiber bundle in the analytic Zariski topology with fiber F_ν

$$F_\nu \simeq \prod_{j=1}^{k} \mathbb{A}_{[0]}^{2^{[\nu_j]}}. \tag{3}$$

We shall deal only with complex analytic spaces Z for which the singular cohomology with compact supports satisifes $\dim_{\mathbb{Q}} H_c^*(W, \mathbb{Q}) < \infty$; in particular the Euler numbers $e(Z)$ are defined. Alexander-Spanier cohomology, singular cohomology and sheaf cohomology with constant coefficients (all with compact supports) are naturally isomorphic to each other on the spaces that we will consider.

Alexander-Spanier's Theory has good properties with respect to closed subsets. We shall need two of these properties. In what follows we assume that W, Y, Z and Z_i are analytic spaces with finite

dimensional rational cohomology with compact supports. Let $Z = \coprod_{i=1}^{n} Z_i$ where $\{Z_i\}$ is a finite collection of mutually disjoint locally closed subsets of Z. Then

$$e(Z) = \sum_{i=1}^{n} e(Z_i). \tag{4}$$

Let $f : W \to Z$ be a continuous map which is locally trivial, with fiber Y, with respect to the stratification Σ_i, i.e. $f^{-1}(Z_i) \simeq \Sigma_i \times Y$ for every $i = 1, \ldots, n$. Then

$$e(W) = e(Y)\, e(Z). \tag{5}$$

It should be now clear what we should be aiming at: *reduce the computation of $e\left(X^{[n]}\right)$ to the computation of $e\left(\widetilde{S_\nu^n X}\right)$ via (4). The numbers $e\left(\widetilde{S_\nu^n X}\right)$ will be computed using (5) and (3). The resulting sum will turn out to be a multinomial expansion.*

2 Statement of the result and the numbers $e\left(S_\nu^{(n)} X\right)$

The following is the main result of this note.

Theorem 2.1 *Let X be a smooth complex surface such that $\dim_{\mathbb{Q}} H_c^*(X, \mathbb{Q}) < \infty$ and with topological Euler characteristic $e(X)$. Let $X^{[n]}$ be the associated Douady spaces of "n-points." The following is a generating function for $e\left(X^{[n]}\right)$:*

$$\sum_{n=0}^{\infty} e\left(X^{[n]}\right) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^{e(X)}. \tag{6}$$

Remark 2.2 The assumptions of the theorem are satisfied for any *algebraic* surface.

The proof will be preceded by some elementary calculations.

The structure of $S_\nu^n X$. Let $\nu = (1, \ldots, 1) \in P(n)$. The stratum $S_{(1,\ldots,1)}^n X$ is the unique Zariski open subset in $X^{(n)}$ belonging to the stratification (1). Let D^n be the big diagonal in the cartesian product X^n. The stratum $S_{(1,\ldots,1)}^n X$ is the quotient of $X^n \setminus D^n$ under the natural free action of the symmetric group over n elements Σ_n. It follows that $S_{(1,\ldots,1)}^n X$ is smooth of complex dimension $2n$. Each stratum $S_\nu^n X$, $\nu \in P(n)$, is built from these basic strata $S_{(1,\ldots,1)}^m X$ for $m \leq n$. Here is how. Let $\nu = (\nu_1, \ldots, \nu_k) \leftrightarrow \{\alpha_1, \ldots, \alpha_n\}$ be the same partition in the two different pieces of notation. There is a natural isomorphism:

$$S_\nu^n X \simeq \left(\prod_{i=1,\, \alpha_i \neq 0}^{n} \left(S_{(1,\ldots,1)}^{\alpha_i} X\right)\right) \setminus \Delta, \tag{7}$$

where Δ is the closed set

$$\Delta := \left\{ (C_1, \ldots, C_l) \in \prod_{i=1,\, \alpha_i \neq 0}^{n} S_{(1,\ldots,1)}^{\alpha_i} X \mid Supp(C_h) \cap Supp(C_j) \neq \emptyset \, for \, some \, h \neq j \right\}.$$

It follows that each stratum $S_\nu^n X$ is smooth, connected and of complex dimension $2k = 2\lambda(\nu)$.

The numbers $e(S_\nu^n X)$. Recalling the convention $0! = 1$, we have the following elementary

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Lemma 2.3 *Let X be a smooth complex analytic surface with $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) < \infty$, $e = e(X)$, $n \in \mathbb{N}^*$ and $P(n) \ni \nu = (\nu_1, \ldots, \nu_k) \leftrightarrow \{\alpha_1, \ldots, \alpha_n\}$. Then*

$$e\left(S_\nu^n X\right) = \frac{1}{\alpha_1! \ldots \alpha_n!} \, e\,(e-1) \ldots (e-(\lambda(\nu)-1)). \tag{8}$$

Proof. We first compute $e\left(S_{(1,\ldots,1)}^m X\right)$ for every m. Since $S_{(1,\ldots,1)}^m X = (X^m \setminus D^m)/\Sigma_m$, and the action is free, we have that, by the multiplicativity of the number e under finite covering maps, $m! \, e\left(S_{1,\ldots,1}^m X\right) = e\left(X^m \setminus D^m\right)$.

We now prove that $e\left(X^m \setminus D^m\right) = e(e-1)\ldots(e-(m-1))$ by induction on m. The case $m = 1$ is trivial. Assume that the above formula is true for $m - 1$ and let us prove it for m. We have a commutative diagram

$$\begin{array}{ccc}
X^m \setminus D^m & \hookrightarrow & (X^{m-1} \setminus D^{m-1}) \times X \\[1em]
\downarrow proj_{1,\ldots,m-1} & \swarrow proj_1 & \\[1em]
X^{m-1} \setminus D^{m-1} & &
\end{array}$$

where the horizontal map is the natural open embedding, the vertical map is the projection to the first $(m-1)$ coordinates and the "south-west" map is the projection onto the first factor. The map $proj_1$ admits $(m-1)$ natural sections $(x_1, \ldots, x_{m-1}) \to (x_1, \ldots, x_{m-1}, x_l)$, where $l = 1, \ldots, m-1$. The images of these sections are pairwise disjoint. Their union is a closed subset of $(X^{m-1} \setminus D^{m-1}) \times X$ isomorphic to $(m-1)$ disjoint copies of $X^{m-1} \setminus D^{m-1}$; the complement of this set is the open subset which we identify with $X^m \setminus D^m$ via the open immersion in the diagram above. The additive and multiplicative properties of the number e and the inductive hypothesis give that

$$\begin{aligned}
e\left(X^m \setminus D^m\right) & = & [\,e(e-1)\ldots(e-(m-1-1))e\,] - [\,e(e-1)\ldots(e-(m-1-1))(m-1)\,] \quad (9) \\
& = & e(e-1)\ldots(e-(m-1)). \quad (10)
\end{aligned}$$

This proves that

$$m! \, e\left(S_{1,\ldots,1}^m X\right) = e\left(X^m \setminus D^m\right) = e(e-1)\ldots(e-(m-1)). \tag{11}$$

As to the case of $S_\nu^n X$ we note that, because of the isomorphism given in (7), this space is the quotient of $X^{\alpha_1+\ldots+\alpha_n=\lambda(\nu)} \setminus D^{\lambda(\nu)}$ under the free action of the group $\Sigma_{\alpha_1} \times \ldots \times \Sigma_{\alpha_n}$ induced by the individual actions of the groups Σ_{α_i} on the factors X^{α_i}. The formula follows from the multiplicativity of Euler characteristics under finite covering maps. $\square$

Remark 2.4 It is an amusing exercise to derive Macdonald formula for the Euler characteristics of symmetric products using Lemma 2.3 and the multinomial expansion; i.e. prove that

$$\sum_{n=0}^{\infty} e\left(X^{(n)}\right) q^n = \left(\frac{1}{1-q}\right)^{e(X)}. \tag{12}$$

The numbers $e(\widetilde{S_\nu^n X})$. By virtue of Lemma 2.3, Theorem 1.1 and (3), the basic additive and multiplicative properties of cohomology with compact support give the following

Lemma 2.5 *Let X be a complex surface with finite Betti numbers. Then*

$$\begin{aligned}
e\left(\widetilde{S_\nu^n X}\right) & = & e\left(\prod_{j=1}^{\lambda(\nu)} \mathbb{A}_{[0]}^{2^{[\nu_j]}}\right) e(S_\nu^n X) = p(\nu_1)\ldots p(\nu_k) \, \frac{1}{\alpha_1! \ldots \alpha_n!} \, e\,(e-1) \ldots (e-(\lambda(\nu)-1)) \\[1em]
& = & p(1)^{\alpha_1} \ldots p(n)^{\alpha_n} \, \frac{1}{\alpha_1! \ldots \alpha_n!} \, e\,(e-1) \ldots (e-(\lambda(\nu)-1)). \quad (13)
\end{aligned}$$

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3 Proof of Theorem 2.1

Proof of Theorem 2.1. By virtue of (2), (4) and Lemma 2.5 we have

$$e\left(X^{[n]}\right) \quad = \quad \sum_{\nu \in P(n)} e\left(\widetilde{S_\nu^n X}\right) \tag{14}$$

$$= \quad \sum_{k=1}^{n} \sum_{\substack{\nu \,\in\, P(n) \\ \lambda(\nu) = k}} \frac{p(1)^{\alpha_1} \dots p(n)^{\alpha_n}}{\alpha_1! \dots \alpha_n!} \, e\,(e-1)\,\dots\,(e-(k-1)). \tag{15}$$

The last summand is the expansion of the r.h.s. of equation (6). Let us check this for $e \geq 0$ from a combinatorial point of view. Set $p(0) := 1$. For $e \geq 0$ the product $\prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^{e(X)}$ can be defined as the limit, in $\mathbb{Q}[[q]]$, of the polynomials $P_n(q) := (\sum_{s=0}^{n} p(s)\, q^s)^e$. The coefficients $c_{m,s}$ of q^s in $P_m(q)$ are the same as the one of the infinite product for every $s \leq n$ and every $m \geq n$. For $n \leq e$ their form is given by the multinomial coefficient expansion in the following way:

$$c_{m,s} \quad = \quad \sum_{\substack{\sum_{i=0}^{n} a_i = e \\ \sum_{i=0}^{n} i\, a_i = n}} \binom{e}{a_0, \dots, a_n} p(0)^{a_0} \dots p(n)^{a_n} \tag{16}$$

$$= \quad \sum_{k=1}^{n} \sum_{\substack{\sum_{j=1}^{n} a_j = e - k \\ \sum_{j=1}^{n} j\, a_j = n}} \binom{e}{k, a_1, \dots, a_n} p(1)^{a_1} \dots p(n)^{a_n} \tag{17}$$

$$= \quad \sum_{k=1}^{n} \sum_{\substack{\sum_{j=1}^{n} a_j = e - k \\ \sum_{j=1}^{n} j\, a_j = n}} \frac{p(1)^{\alpha_1} \dots p(n)^{\alpha_n}}{\alpha_1! \dots \alpha_n!} \, e\,(e-1)\,\dots\,(e-(k-1)), \tag{18}$$

which coincides with $e\left(X^{[n]}\right)$. The formula is still correct for $n > e$ (many summands are zero). The case $e = 0$ is trivial. We leave the case $e < 0$ to the reader. $\qquad\square$

Remark 3.1 The generating function of Theorem 2.1 exhibits modular behavior (cf. [9]).

Remark 3.2 If X is not algebraic, then the approach in [4] cannot be used to prove Theorem 2.1. The paper [10] restricts itself to the algebraic context. However, that restriction is unnecessary. This is remarked in [5], whose main purpose is to prove the necessary decomposition theorem in the form of an explicit quasi-isomorphism of complexes.

Remark 3.3 For X algebraic the numbers $e\left(X^{[n]}\right)$ and the orbifold Euler characteristics $e(X^n, \Sigma_n)$ coincide. Similarly, we see that because of Theorem 2.1 this "coincidence" occurs for any complex surface, i.e. not necessarily algebraic. This is explained by the fact that the groups $K(X^{[n]}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the Σ_n-Equivariant K-Theory of X^n, $K_{\Sigma_n}(X^n) \otimes_{\mathbb{Z}} \mathbb{Q}$, are *naturally* isomorphic. This fact is due to B. Bezrukavnikov and V. Ginzburg [2] and, independently, to L. Migliorini and myself [5].

Question 3.4 The numbers $e\left(X^{[n]}\right)$ are invariant in the class of complex analytic smooth surfaces homotopically equivalent to X. Is $X^{[n]}$ a topological invariant of X? To be precise, let X and Y be complex surfaces homeomorphic to each other; is $X^{[n]}$ homeomorphic to $Y^{[n]}$? Is $X^{[n]}$ an invariant for the differentiable structure of X?

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